CT Convolution

For DT $\quad y[n] = x[n] * h[n] \triangleq \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$

Convolution Integral

$$y(t) = x(t) \star h(t) \triangleq \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau$$

Interpretation

$$h(\tau) \xrightarrow{\text{Flip}} h(-\tau)$$

$$h(-\tau) \xrightarrow{\text{Shift}} h(t-\tau)$$

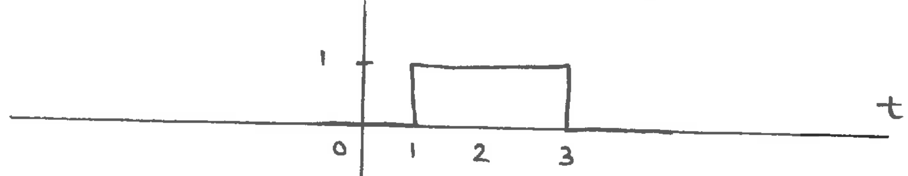
$$h(t-\tau) \xrightarrow{\text{Multiply}} x(\tau)h(t-\tau)$$

$$x(\tau)h(t-\tau) \xrightarrow{\text{Integrate}} \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau$$

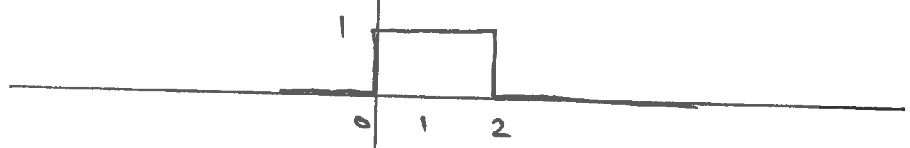
$$x(t) \longrightarrow \boxed{\begin{matrix}\text{LTI} \\ \text{CT System} \\ h(t)\end{matrix}} \longrightarrow y(t) = x(t) * h(t)$$

CT Convolution Example

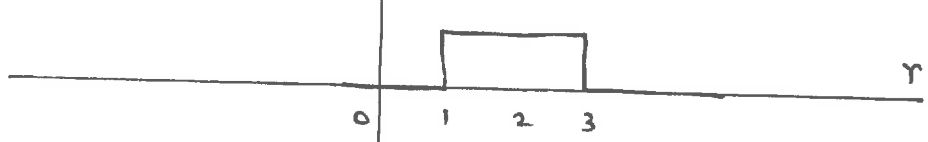
$x(t) \qquad x(t) = u(t-1) - u(t-3)$



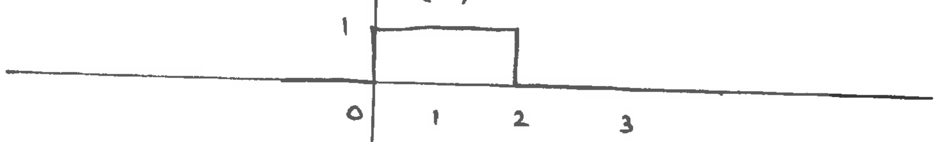
$h(t) = u(t) - u(t-2)$



$x(\tau)$



$h(\tau)$



$h(\tau + t)$

$h(-\tau + t) = h(t - \tau)$

For $t < 1$ There is no overlap, $y(t) = 0$

For $1 \leq t \leq 3$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{1}^{t} (1)(1)\, d\tau = \left| \tau \right|_{1}^{t} = t - 1$$

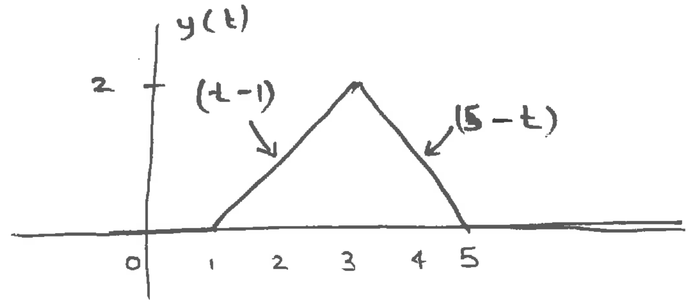
For $3 \leq t \leq 5$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{t-2}^{3} (1)(1)\, d\tau$$

$$= \left| \tau \right|_{t-2}^{3} = 5 - t$$

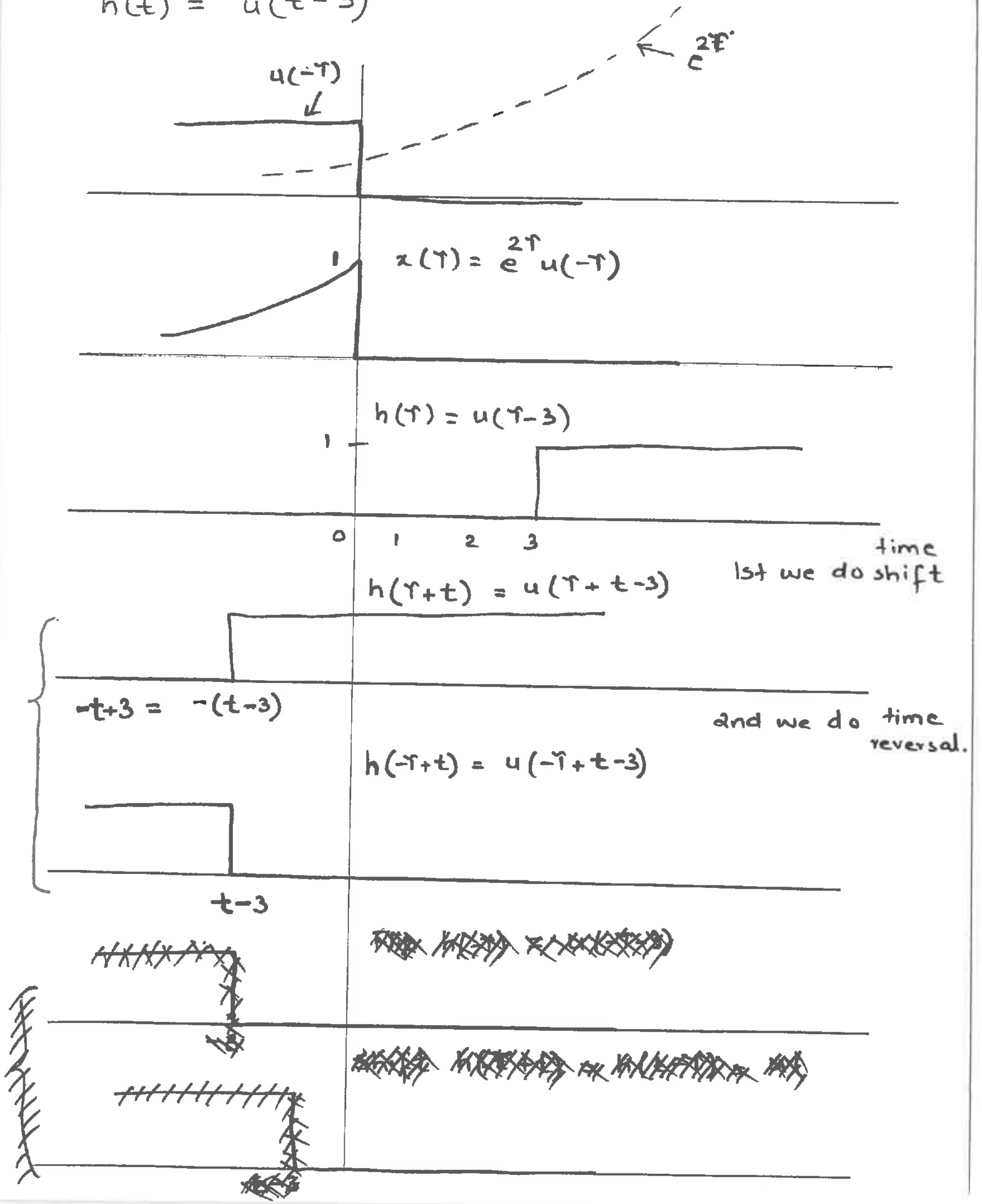
For $t \geq 5$ no overlap, $y(t) = 0$.



CT Convolution Example 2

$$x(t) = e^{2t}u(-t)$$

$$h(t) = u(t-3)$$



For $t-3 < 0$

$$\Rightarrow \quad t < 3$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{t-3} e^{2\tau}\, d\tau$$

$$= \frac{1}{2}\left| e^{2\tau} \right|_{-\infty}^{t-3} = \frac{1}{2}\left(e^{2(t-3)} - e^{-\infty} \right)$$

$$= \frac{1}{2} e^{2(t-3)}$$

For $t-3 \geq 0$

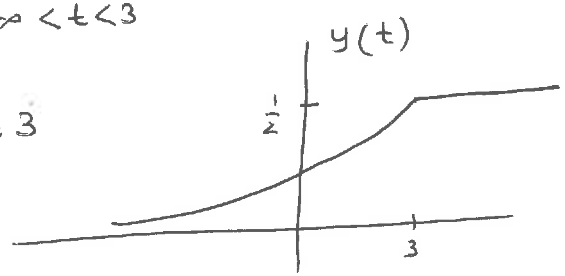
$$\Rightarrow \quad t \geq 3$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{0} e^{2\tau}\, d\tau$$

$$= \frac{1}{2}\left| e^{2\tau} \right|_{-\infty}^{0} = \frac{1}{2}\left(e^{0} - e^{-\infty} \right) = \frac{1}{2}(1-0) = \frac{1}{2}$$

$$y(t) = \begin{cases} \frac{1}{2} e^{2(t-3)} & -\infty < t < 3 \\ \frac{1}{2} & t \geq 3 \end{cases}$$



Properties of Impulse & Unit Step Functions

(1) How to visualize $\delta(t)$

(2) Relationship between $u(t)$ and $\delta(t)$

(3) Convolution with $\delta(t)$

(4) Sifting property of $\delta(t-t_0)$

Note :

$x(t) * h(t) = h(t) * x(t)$ → Commutative property.

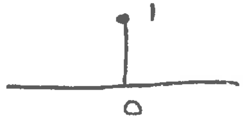
$x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t)$ → Distributive property

$x(t) * (h_1(t) * h_2(t)) = (x(t) * h_1(t)) * h_2(t)$ → Associative property.

Properties of $\delta(t)$

DT unit Impulse

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$



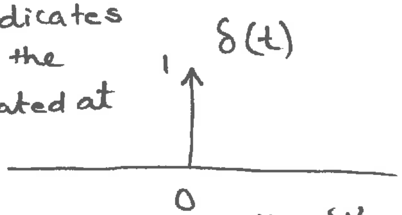
CT unit Impulse is defined by following pair of equations

$$\delta(t) = 0 \text{ for } t \neq 0 \quad \& \quad \int_{-\infty}^{\infty} \delta(t)\, dt = 1.$$

impulse is zero everywhere, except at origin.

total area under the unit impulse is unity.

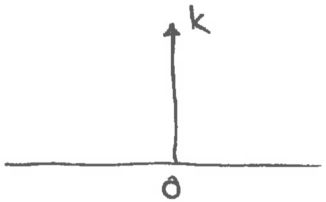
arrow at $t=0$ indicates that the area of the impulse is concentrated at $t=0$.



Dirac delta function.

The height of the arrow and the '1' next to it represent the area of the impulse.

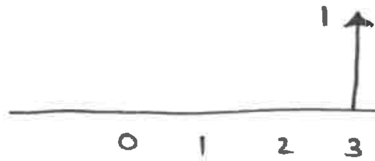
Sketch $k\delta(t)$



Scaled Impulse

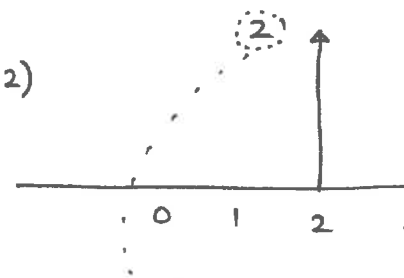
$$\delta(at) = \frac{1}{a}\delta(t) \quad , \quad a > 0 \Rightarrow \text{time-scaling property.}$$

$\delta(t-3)$



$$\int_{-\infty}^{\infty} \delta(t-3)\, dt = 1$$

$2\delta(t-2)$



$$\int_{-\infty}^{\infty} 2\delta(t-2)\, dt = 2$$

This is indicating area under the impulse function.

CT System Properties – Sifting Property (cont'd)

Note that $x(t) \star \delta(t - t_0)$ is different from $x(t)\,\delta(t - t_0)$. The expression on the left is convolution of two signals and the expression on the right is pointwise multiplication.

- Convolution

$$\begin{aligned} x(t) \star \delta(t - t_0) &= \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau - t_0)\, d\tau \\ &= x(t - t_0) \end{aligned}$$

- Pointwise multiplication

$$x(t)\,\delta(t - t_0) = x(t_0)\,\delta(t - t_0)$$

which is a scaled impulse response.

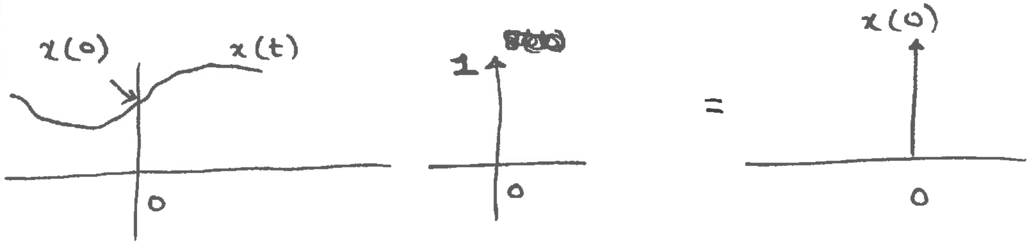
Sifting Property of Impulse

(2) $$\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0)$$

(1) $$x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$$

pointwise multiplication

$$x(t)\,\delta(t) = x(0)\,\delta(t)$$



(3) $$x(t) * \delta(t-t_0) = x(t-t_0)$$

convolution

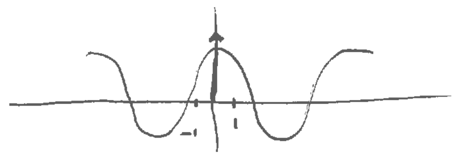
convolving a signal with an impulse shifts the signal to the location of the impulse.

Examples

$$\int_{-\infty}^{\infty} x(t)\,\delta(t)\,dt = x(0)$$

$$\int_{-\infty}^{\infty} x(t)\,\delta(t-5)\,dt = x(5)$$

$$\int_{-1}^{1} \cos t\,\delta(t)\,dt = \cos 0 = 1$$

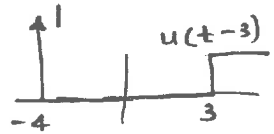


$$\int_{1}^{3} \cos t\,\delta(t)\,dt = 0$$ (since $\delta(t) = 0$ between $1 \leq t \leq 3$)

$$\int_{1}^{3} \cos t\,\delta(t-2)\,dt = \cos 2$$ (since $\delta(t-2) = 1$ at $t = 2$ and this is inside the integration range)

$$\int_{-\infty}^{\infty} e^{-4(t-1)}\,\delta(t-2)\,dt = e^{-4(2-1)} = e^{-4}$$

$$\int_{-\infty}^{\infty} u(t-3)\,\delta(t+4) = 0$$

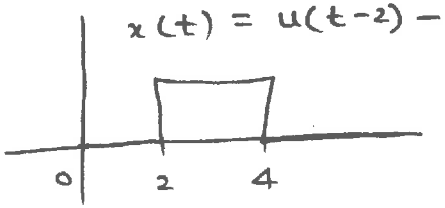


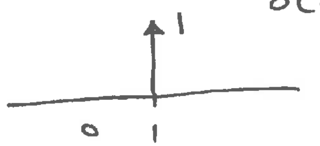
Convolution with Impulses

$$x(t) * \delta(t-t_0) = x(t-t_0)$$

Example

$x(t) = u(t-2) - u(t-4)$ $\quad * \quad$ $\delta(t-1)$





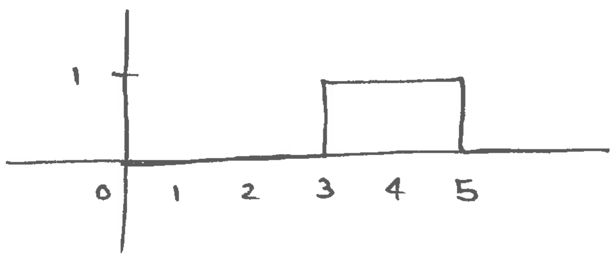
$$x(t) * \delta(t-1) = \big(u(t-2) - u(t-4)\big) * \delta(t-1)$$

distributive property of convolution.

$$= u(t-2) * \delta(t-1) - u(t-4) * \delta(t-1)$$

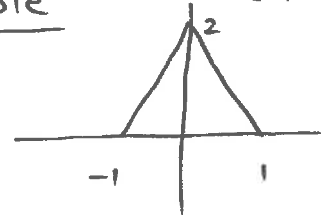
$$= u(t-1-2) - u(t-1-4)$$

$$= u(t-3) - u(t-5)$$

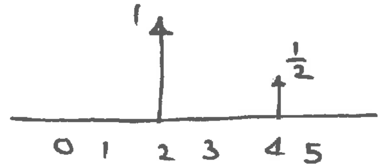


Example

$x(t)$



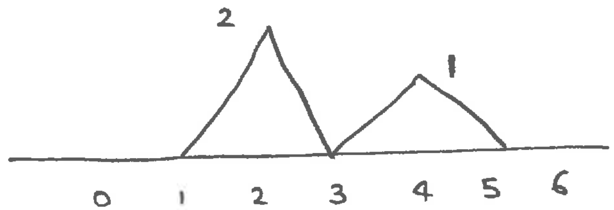
$*$



$$x(t) * \left\{\delta(t-2) + \tfrac{1}{2}\delta(t-4)\right\}$$

$$= x(t) * \delta(t-2) + \frac{1}{2}\, x(t) * \delta(t-4)$$

$$= x(t-2) + \frac{1}{2}\, x(t-4)$$



Example

$$h(t) = \delta(t+1) - \delta(t) + 2\delta(t-2)$$

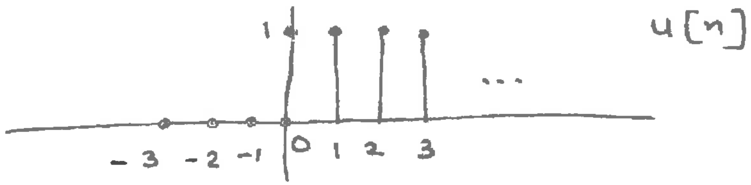
$$x(t) = e^{-t} u(t)$$

$$y(t) = x(t) * h(t)$$

$$= e^{-(t+1)} u(t+1) - e^{-t} u(t) + 2e^{-(t-2)} u(t-2)$$

From L05 notes

Representing DT signals using unit impulse functions



$$u[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \ldots \qquad \delta[n] = \begin{cases} 1 & n=0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \sum_{k=0}^{\infty} \delta[n-k]$$

unit-step = superposition of delayed impulses

$$\delta[n] = u[n] - u[n-1]$$

unit impulse = first difference of DT unitstep

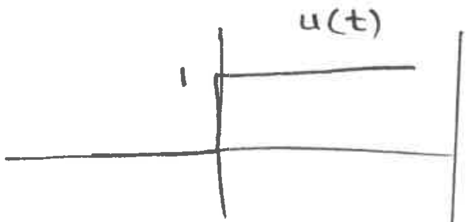
$$\boxed{u[n] = \sum_{k=-\infty}^{n} \delta[k]}$$

DT unit step = running sum of the unit ~~im~~ impulse.

Relationship with $u(t)$

Unit Step Function

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$



CT unit step is the running integral of the unit impulse

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$

$$\delta(t) = \frac{d}{dt} u(t)$$

CT unit impulse can be thought of as the 1st derivative of the CT unit-step.

Another way to relate $u(t)$ & $\delta(t)$

$$u(t) = \int_{0}^{\infty} \delta(t - \sigma)\, d\sigma$$

Since area of $\delta(t-\sigma)$ is concentrated at point $\sigma = t$, the integral is 0 for $t < 0$ and 1 for $t > 0$.

Section 1.4.2 (pages 32 – 36 in textbook)

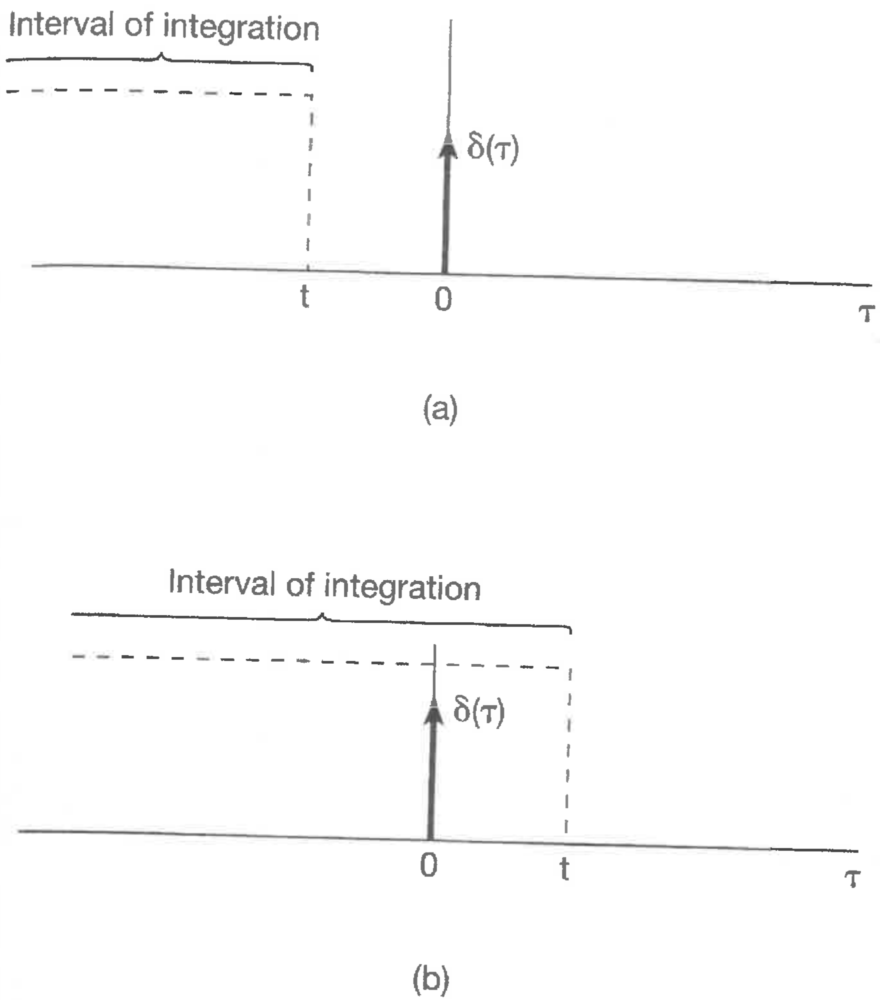


Figure 1.37 Running integral given in eq. (1.71): (a) $t < 0$; (b) $t > 0$.

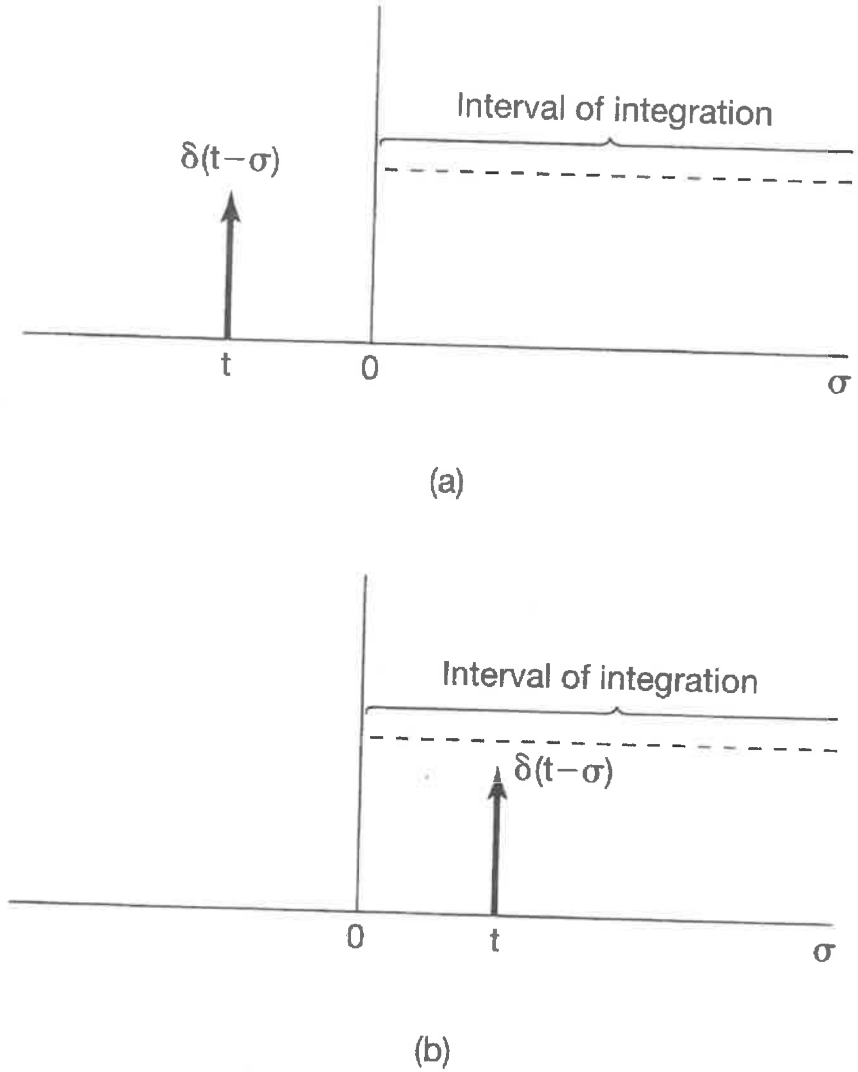


Figure 1.38 Relationship given in eq. (1.75): (a) $t < 0$; (b) $t > 0$.

We can represent this as:

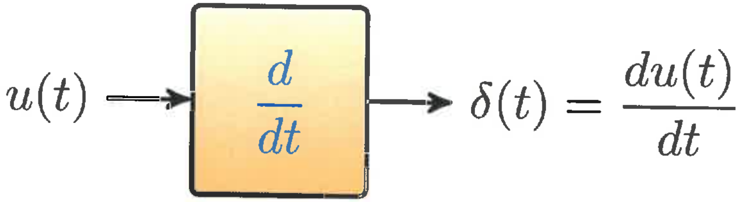


Fig: Differentiator with Unit Step Input

where the $\frac{d}{dt}$ inside the box is not the impulse response but denotes an operator. (cont'd)